Quasi-periodic perturbations within the reversible context 2 in KAM theory

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Abstract

The paper consists of two sections. In Section 1, we give a short review of KAM theory with an emphasis on Whitney smooth families of invariant tori in typical Hamiltonian and reversible systems. In Section 2, we prove a KAM-type result for non-autonomous reversible systems (depending quasi-periodically on time) within the almost unexplored reversible context 2. This context refers to the situation where $\dim \operatorname{Fix} G < \frac{1}{2} \operatorname{codim} \mathcal{T}$, here $\operatorname{Fix} G$ is the fixed point manifold of the reversing involution G and $\mathcal{T}$ is the invariant torus one deals with.

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1. KAM theory from a bird's eye view

KAM (Kolmogorov–Arnold–Moser) theory is the theory of quasi-periodic motions (i.e. conditionally periodic motions with incommensurable frequencies) in non-integrable dynamical systems. The phase curves of such motions fill up densely invariant tori (so called quasi-periodic invariant tori) in the phase space. In turn, these tori are usually organized into complicated hierarchical structures consisting of tori of different dimensions. However, the "building blocks" of such structures are Whitney smooth Cantor-like families of invariant tori rather than individual tori. We refer the reader to e.g. [9, 10] for a precise definition of Whitney smooth families of invariant tori (to be brief, a Whitney smooth function on a closed set in $\mathbb{R}^\ell$ is a function extendible to a smooth function on a neighborhood of this set). The properties of Whitney smooth families of invariant tori depend strongly on the symmetries preserved by the system in question.

1.1. Hamiltonian systems

As our first example, consider autonomous Hamiltonian vector fields on a finite dimensional manifold equipped with an *exact* symplectic 2-form. In the Hamiltonian KAM theory, the following easy observation is of crucial importance.

Herman's lemma. *Any quasi-periodic invariant torus of a Hamiltonian system is isotropic provided that the symplectic form is exact. In particular, the dimension of such a torus does not exceed the number of degrees of freedom.*

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Recall that a submanifold $\mathcal{L}$ of a symplectic manifold is said to be *isotropic* if the restriction of the symplectic form to $\mathcal{L}$ vanishes, i.e., the tangent space $T_a\mathcal{L}$ is contained in its skew orthogonal complement for each point $a \in \mathcal{L}$. Herman himself [16, 17] proved the lemma above in a certain particular case but the general case [10, 33] is not harder at all (in fact, this lemma goes back to Moser [23, pp. 157–158]). Herman's lemma can be carried over to locally Hamiltonian systems, i.e. to Hamiltonian systems with multi-valued Hamilton functions (for instance, the system $\dot{x} = 0$, $\dot{y} = 1$ on a cylinder with coordinates $x \in \mathbb{S}^1$, $y \in \mathbb{R}$ and the symplectic form $dx \wedge dy$ is locally Hamiltonian: it is afforded by the Hamilton function x with values in $\mathbb{S}^1$ rather than in $\mathbb{R}$).

Remark 1. Of course, invariant tori of dimensions 0 (equilibria) and 1 (periodic trajectories) are always isotropic whether or not the symplectic form is exact. The same is valid for invariant 2-tori. Indeed, consider an invariant manifold $\mathcal{L}$ of a Hamiltonian flow with Hamilton function $\mathcal{H}$. Suppose that $\mathcal{H}|_{\mathcal{L}} = \text{const}$ and that almost all the points of $\mathcal{L}$ are not equilibria. It is not hard to verify that $\mathcal{L}$ is isotropic if $\dim\mathcal{L} = 2$ and coisotropic if $\text{codim}\,\mathcal{L} = 2$ (regardless of whether the symplectic form is exact).

Recall that a submanifold $\mathcal{L}$ of a symplectic manifold is said to be *coisotropic* if the tangent space $T_a\mathcal{L}$ contains its skew orthogonal complement for each point $a \in \mathcal{L}$.

The following concepts are also of principal importance in KAM theory.

Definition 1. Let an invariant n-torus $\mathcal{T}$ of some flow on an $(n + \ell)$-dimensional manifold carry conditionally periodic motions with frequency vector $\omega \in \mathbb{R}^n$. This torus is said to be *reducible* (or *Floquet*) if in a neighborhood of $\mathcal{T}$, there exists a coordinate frame $x \in \mathbb{T}^n$, $w \in \mathcal{O}_\ell(0)$ in which the torus $\mathcal{T}$ itself is given by the equation $\{w = 0\}$ and the dynamical system takes the *Floquet form* $\dot{x} = \omega + \mathrm{O}\big(|w|\big)$, $\dot{w} = Lw + \mathrm{O}\big(|w|^2\big)$ with an x-independent matrix $L \in \mathfrak{gl}(\ell, \mathbb{R})$. This matrix (not determined uniquely) is called the *Floquet matrix* of the torus $\mathcal{T}$, and its eigenvalues are called the *Floquet exponents* of $\mathcal{T}$.

Here and henceforth, $\mathbb{T}^n = (\mathbb{S}^1)^n = (\mathbb{R}/2\pi\mathbb{Z})^n$ is the standard n-torus, while $\mathcal{O}_\ell(a)$ denotes an unspecified neighborhood of a point $a \in \mathbb{R}^\ell$.

In other words, an invariant torus is reducible if the variational equations along this torus can be reduced to a form with constant coefficients.

The essence of the Hamiltonian KAM theory can now be formulated as follows.

Hamiltonian KAM paradigm (for exact symplectic forms). *In a* typical *Hamiltonian system with $M \geqslant 1$ degrees of freedom (in the case of an exact symplectic 2-form), there are*

- *isolated equilibria,*

- *smooth one-parameter families of closed trajectories (one trajectory per energy value),*

- *and Whitney smooth Cantor-like n-parameter families of isotropic invariant n-tori carrying quasi-periodic motions with strongly incommensurable (e.g. Diophantine) frequencies for each $2 \leqslant n \leqslant M$.*

These tori can be either reducible or non-reducible. The Floquet exponents of a reducible invariant n-torus include value 0 of multiplicity n (for $1 \leqslant n \leqslant M$) whereas the remaining $2M - 2n$ Floquet exponents come in pairs $\lambda, -\lambda$ (for $0 \leqslant n \leqslant M - 1$).

The word "typical" here means that Hamiltonian systems with the properties indicated constitute an open set (to be more precise, a set with non-empty interior) in the functional space of all the Hamiltonian systems with M degrees of freedom. The meaning of this word in the sequel will be similar.

Isotropic invariant n-tori in a Hamiltonian system with M degrees of freedom are said to be *Lagrangian* for $n = M$ and *lower dimensional* for $n < M$.

By now, Whitney smooth Cantor-like families of isotropic invariant tori in Hamiltonian systems have been thoroughly explored, especially in the reducible case. The reader is referred to [2, 9–12, 25, 30, 33] for surveys, precise statements, and bibliographies. Here we will confine ourselves with the following remark. In KAM theory, there are known phenomena leading to a decrease or increase in the dimension of invariant tori.

One of the "lowering" phenomena is destruction of resonant tori. Consider a partially integrable Hamiltonian system possessing a smooth n-parameter family of isotropic invariant n-tori ($n \geqslant 2$) carrying conditionally periodic motions with frequency vectors $\omega(\mu)$, where μ is the parameter of the family. Let $1 \leqslant r \leqslant n-1$. Typically, this n-parameter family of n-tori contains an $(n-r)$-parameter subfamily of tori whose frequencies satisfy r independent *fixed* resonance relations $\langle q^{(i)}, \omega(\mu) \rangle = 0$, $q^{(i)} \in \mathbb{Z}^n$, $1 \leqslant i \leqslant r$ (here and henceforth, the angle brackets denote the standard inner product). Then, under a generic Hamiltonian perturbation, this smooth $(n-r)$-parameter subfamily of resonant n-tori gives rise to a finite collection of Whitney smooth Cantor-like $(n-r)$-parameter families of isotropic quasi-periodic invariant $(n-r)$-tori (so called Treshchëv tori, see [2, 10, 12, 33] for surveys and references). Of course, for $r = n - 1$, these 1-parameter families of closed trajectories (called Poincaré trajectories) are smooth rather than Cantor-like. In fact, break-up of resonant tori has been studied by now in the Lagrangian case only (for n equal to the number M of degrees of freedom).

One of the "raising" phenomena is excitation of elliptic normal modes. Consider an n-parameter family of reducible isotropic invariant n-tori in a Hamiltonian system with $M > n$ degrees of freedom. Let $\omega(\mu)$ be the frequency vectors of the tori, where μ is the parameter of the family. Suppose that among the non-zero Floquet exponents of each of these tori, there are r pairs ($1 \leqslant r \leqslant M - n$) of purely imaginary numbers $\pm i\beta_1(\mu), \ldots, \pm i\beta_r(\mu)$ that depend smoothly on μ (in the Whitney sense for $n \geqslant 2$). The remaining $2(M - n - r)$ non-zero Floquet exponents are also allowed to be purely imaginary. Then, generically, in a neighborhood of this n-parameter family of n-tori, there is an $(n+r)$-parameter family of reducible isotropic invariant $(n+r)$-tori. The frequencies of these tori are close to $\omega_1(\mu), \ldots, \omega_n(\mu), \beta_1(\mu), \ldots, \beta_r(\mu)$. One says that the Floquet exponents $\pm i\beta_1(\mu), \ldots, \pm i\beta_r(\mu)$ "excite", see again [2, 10, 12, 33] for surveys and references. For instance, a neighborhood of a generic elliptic equilibrium ($n = 0$, $r = M$) contains a Whitney smooth Cantor-like M-parameter family of Lagrangian invariant M-tori. The simplest example is $n = 0$, $r = M = 1$: an elliptic equilibrium of a planar Hamiltonian system is surrounded by a smooth one-parameter family of periodic trajectories (energy levels). In the case where $n = 0$, $r = 1$, $M > 1$, similar smooth one-parameter families of closed trajectories are called Lyapunov families.

We emphasize that, starting from n-parameter families of invariant n-tori, we obtain $(n \pm r)$-parameter families of invariant $(n \pm r)$-tori, in a complete agreement with the paradigm above.

In the case of non-exact symplectic forms, there become possible coisotropic invariant n-tori with $M + 1 \leqslant n \leqslant 2M - 1$ (such tori are studied in Parasyuk's theory and are sometimes said to be *higher dimensional*) as well as so called *atropic* invariant n-tori (i.e., tori that are neither

isotropic nor coisotropic) with $3 \leqslant n \leqslant 2M-3$. As before, M is the number of degrees of freedom. Coisotropic and a fortiori atropic invariant tori have been explored much worse than isotropic ones, we refer the reader to [9, 10, 12, 33, 37] for discussions and relevant bibliographies. Atropic quasi-periodic invariant tori of dimensions 2 and $2M-2$ cannot exist according to Remark 1.

Remark 2. In [25], we introduced the concept of s-*exact* symplectic forms: a symplectic 2-form is said to be s-*exact* ($s \geqslant 1$) if its sth exterior power is exact (it is clear that an s-exact symplectic form is also s'-exact for any $s' > s$). Seizing the opportunity, we now make an important comment to this concept. Consider a $2M$-dimensional connected manifold $\mathcal{M}$. If $\mathcal{M}$ is *closed* (i.e. compact without boundary) then it cannot admit even M-exact symplectic forms since the Mth exterior power of a symplectic form on $\mathcal{M}$ is a volume element. On the other hand, if $\mathcal{M}$ is *open* (i.e. either noncompact or with a non-empty boundary) and carries a nondegenerate 2-form then it also admits an exact symplectic form. In fact, according to Gromov's theorem (see e.g. [20, Section 7.3]), every homotopy class of nondegenerate 2-forms on $\mathcal{M}$ can be represented by a symplectic form representing any prescribed cohomology class in $H^2(\mathcal{M}, \mathbb{R})$. In particular, any manifold $\mathbb{R}^k \times \mathbb{T}^l$ with $k+l$ even and $k \geqslant 1$ admits an exact symplectic form (this is of course obvious for $k \geqslant l$ but not obvious at all for $0 < k < l$).

1.2. Reversible systems

Our second and main example is reversible vector fields. Consider a finite dimensional connected manifold $\mathcal{M}$ and a smooth involution $G : \mathcal{M} \to \mathcal{M}$ (G^2 is the identity transformation).

Definition 2. A vector field V on $\mathcal{M}$ is said to be *reversible* with respect to involution G (or G-*reversible*) if $\operatorname{Ad} G(V) = TG(V \circ G) = -V$ or, equivalently, if the function $t \mapsto G\big(a(-t)\big)$ is a solution of the equation $\dot{a} = V(a)$, $a \in \mathcal{M}$, whenever $\mathbb{R} \ni t \mapsto a(t)$ is.

For instance, the Newtonian equations of motion $\ddot{\mathbf{w}} = F(\mathbf{w}, \dot{\mathbf{w}})$, $\mathbf{w} \in \mathbb{R}^\ell$, are reversible with respect to the phase space involution $G : (\mathbf{w}, \dot{\mathbf{w}}) \mapsto (\mathbf{w}, -\dot{\mathbf{w}})$ if and only if the forces F are even in the velocities $\dot{\mathbf{w}}$ (e.g. are independent of $\dot{\mathbf{w}}$). The papers [18, 26] present general surveys of the theory of reversible systems with extensive bibliographies.

If a submanifold $\mathcal{L} \subset \mathcal{M}$ is invariant under a G-reversible flow, so is its "mirror" image $G(\mathcal{L})$. If $G(\mathcal{L}) \neq \mathcal{L}$ then, as a rule, the dynamics near each of the two invariant manifolds $\mathcal{L}$ and $G(\mathcal{L})$ considered separately exhibits no special features. Therefore, while speaking of an invariant manifold $\mathcal{L}$ of a G-reversible system, one usually assumes $\mathcal{L}$ to be invariant not only under the flow itself but also under the reverser G. In the sequel, the words "an invariant torus (in particular, an equilibrium or periodic trajectory) of a reversible system" will always mean a torus invariant under both the flow and the reversing involution.

The crucial role in the dynamics of G-reversible systems on $\mathcal{M}$ is played by the fixed point set $\operatorname{Fix} G = \big\{ a \in \mathcal{M} \mid G(a) = a \big\}$. This set is a submanifold of the same smoothness class as the involution G itself (a very particular case of Bochner's theorem [4, 21], see also [13]). The fixed point manifold $\operatorname{Fix} G$ can well be empty or consist of several connected components of different dimensions. Extensive information on the fixed point submanifolds of involutions of various manifolds is presented in e.g. the books [4, 13], see also the articles [25, 30, 38].

As is widely known, there is a deep similarity between reversible and Hamiltonian dynamics [1, 3, 18, 26–28]. In particular, many fundamental results of the Hamiltonian KAM theory possess reversible counterparts. The reversible analog of Herman's lemma is the following (also very easy) statement.

Standard reflection lemma ([9, 10, 27, 39]). *In any quasi-periodic invariant n-torus $\mathcal{T}$ of a G-reversible flow, one can introduce a coordinate frame $\varphi \in \mathbb{T}^n$ in which the dynamics on $\mathcal{T}$ takes the form $\dot{\varphi} = \omega$ ($\omega \in \mathbb{R}^n$ being the frequency vector of $\mathcal{T}$) and the restriction of G to $\mathcal{T}$ takes the form $G|_{\mathcal{T}} : \varphi \mapsto -\varphi$. In particular, $\mathcal{T} \cap \operatorname{Fix} G$ consists of 2^n points φ with φ_i equal to either 0 or π, $1 \leqslant i \leqslant n$, and the dimension of any connected component of $\operatorname{Fix} G$ having a non-empty intersection with $\mathcal{T}$ does not exceed $\operatorname{codim} \mathcal{T}$. Moreover, in a smooth (or Whitney smooth) family of quasi-periodic invariant tori, the coordinate φ can be chosen to depend smoothly (respectively Whitney smoothly) on the torus.*

This lemma can be carried over to *weakly reversible* systems, i.e. systems reversed by phase space diffeomorphisms G that are not necessarily involutions [1, 3, 27, 39]. The definition of weakly G-reversible vector fields V has the form $\operatorname{Ad} G(V) = TG(V \circ G^{-1}) = -V$.

In the sequel, we will consider only involutions $G : \mathcal{M} \to \mathcal{M}$ for which $\operatorname{Fix} G \neq \varnothing$ and all the connected components of $\operatorname{Fix} G$ are of the same dimension, so that $\dim \operatorname{Fix} G$ is well defined (this is the case for almost all the reversible systems encountered in practice). We will write $\dim \operatorname{Fix} G = P$ and $\operatorname{codim} \operatorname{Fix} G = Q$, so that $\dim \mathcal{M} = P + Q$. According to the standard reflection lemma, if a G-reversible system admits a quasi-periodic invariant n-torus then $n \leqslant Q$.

To formulate the reversible counterpart of the Hamiltonian KAM paradigm, one has to deal with smooth *families of reversible systems* (depending on an external s-dimensional parameter ν) rather than with individual reversible systems (corresponding to the case where $s = 0$). The reversible analog of the Hamiltonian KAM paradigm is the following statement.

Reversible KAM paradigm. *In the product of the phase space and the parameter space of a typical s-parameter family of G-reversible systems with $\dim \operatorname{Fix} G = P$ and $\operatorname{codim} \operatorname{Fix} G = Q$, there are*

- *smooth $(P - Q + s)$-parameter families of equilibria (for $s \geqslant \max\{Q - P, 0\}$),*

- *smooth $(P - Q + s + 1)$-parameter families of closed trajectories (for $s \geqslant \max\{Q - P - 1, 0\}$),*

- *and Whitney smooth Cantor-like $(P - Q + s + n)$-parameter families of invariant n-tori carrying quasi-periodic motions with strongly incommensurable (e.g. Diophantine) frequencies for each $2 \leqslant n \leqslant Q$ (for $s \geqslant \max\{Q - P - n + 1, 0\}$).*

These tori can be either reducible or non-reducible. The Floquet exponents of a reducible invariant n-torus include value 0 of multiplicity $|P - Q + n|$ (for $P \neq Q - n$) whereas the remaining $2 \min\{P, Q - n\}$ Floquet exponents come in pairs $\lambda, -\lambda$ (for $\min\{P, Q - n\} \geqslant 1$).

In the case where $P > 0$ and $Q > n$, the invariant n-tori in question are said to be *lower dimensional*.

Let us explain $|P - Q + n|$ zero eigenvalues of the Floquet matrix here. Consider a matrix L anti-commuting with a fixed involutive matrix $\mathcal{K} \in \operatorname{GL}(\mathcal{P} + \mathcal{Q}, \mathbb{R})$ [$\mathcal{K}^2$ is the identity matrix], the eigenvalues 1 and -1 of $\mathcal{K}$ being of multiplicities $\mathcal{P}$ and $\mathcal{Q}$, respectively. If $\mathcal{P} \neq \mathcal{Q}$ then 0 is an eigenvalue of L of multiplicity at least $|\mathcal{P} - \mathcal{Q}|$ [27, 29]. The remaining $\mathcal{P} + \mathcal{Q} - |\mathcal{P} - \mathcal{Q}| = 2 \min\{\mathcal{P}, \mathcal{Q}\}$ eigenvalues of L come in pairs $\lambda, -\lambda$ and are generically other than zero [27, 29].

Remark 3. One cannot hope to encounter an isolated quasi-periodic invariant torus of dimension $n \geqslant 2$ in a generic dissipative, volume preserving, Hamiltonian, or reversible system (or in a

generic family of systems). Of course, generic systems may admit isolated invariant n-tori with $n \geqslant 2$, but such tori would not carry conditionally periodic motions (the induced dynamics would be "phase-locked"). Therefore, for $(P-Q+s+n)$-parameter families of invariant tori of dimensions $n \geqslant 2$ in the reversible KAM paradigm, one has $P-Q+s+n \geqslant 1$, i.e. $s \geqslant Q-P-n+1$ rather than $s \geqslant Q-P-n$.

Similarly to the Hamiltonian context, one may consider destruction of resonant tori. Starting with a smooth $(P-Q+s+n)$-parameter family of invariant n-tori carrying conditionally periodic motions, one would obtain Whitney smooth Cantor-like $(P-Q+s+n-r)$-parameter families of quasi-periodic invariant $(n-r)$-tori (for $1 \leqslant r \leqslant n-1$ and $P-Q+s+n-r \geqslant 1$; it suffices to require $P-Q+s+n-r = P-Q+s+1 \geqslant 0$ for $r=n-1$). Excitation of elliptic normal modes makes sense as well. Starting with a $(P-Q+s+n)$-parameter family of reducible invariant n-tori, one expects to obtain $(P-Q+s+n+r)$-parameter families of invariant $(n+r)$-tori for $1 \leqslant r \leqslant \min\{P, Q-n\}$.

However, while the Hamiltonian KAM paradigm has been proven completely by now, one cannot say the same about the reversible KAM paradigm. Consider a quasi-periodic invariant n-torus $\mathcal{T}$ of a G-reversible system with $\dim \operatorname{Fix} G = P$ and $\operatorname{codim} \operatorname{Fix} G = Q$. For this torus, we have two non-negative "characteristic numbers" P and $Q-n$. The codimension of $\mathcal{T}$ is their sum $P+Q-n$.

Definition 3. The situation where

$$P \geqslant Q-n \iff \dim \operatorname{Fix} G \geqslant \tfrac{1}{2} \operatorname{codim} \mathcal{T}$$

is called *the reversible context 1*. The opposite situation where

$$P < Q-n \iff \dim \operatorname{Fix} G < \tfrac{1}{2} \operatorname{codim} \mathcal{T}$$

is called *the reversible context 2*.

It turns out that by now, almost all the reversible KAM theory has been devoted exclusively to the reversible context 1. This context has been nearly as developed as the Hamiltonian KAM theory. Surveys, precise statements of the theorems, and bibliographies are given in e.g. [3, 8–10, 24, 25, 27, 28, 30–32], the reader is also referred to [5–7, 19, 34–36, 40–42] for some important results obtained after the review [32]. By the way, the reversible context 1 requires an external parameter only in the case where $n \geqslant 2$ and $P = Q-n$ (so that $s \geqslant 1$). For other values of P, Q, and n within the reversible context 1, one may consider individual systems ($s = 0$).

The reversible context 2 was first described in [9, 10] where the paradigm for this context was formulated as a conjecture. The paradigm for the reversible context 1 was given in [9, 10] separately (as we saw above, the paradigms for both the contexts can be unified). The task of developing the reversible KAM theory in context 2 was listed (as problem 9) among the ten problems of the classical KAM theory in the note [37]. The first result in the reversible context 2 was obtained no earlier than in 2011 [38]. It concerned the "*extreme*" reversible context 2 where $P = 0$. Lower dimensional reducible invariant tori in the reversible context 2 (with $0 < P < Q-n$) were treated in [39]. Both the papers [38, 39] examine only analytic families of invariant tori (in the analytic category, i.e., under the assumption that the reversing involution, the vector fields themselves and their families are analytic) in the presence of many

external parameters. To the best of the author's knowledge, all the KAM theory for the reversible context 2 is currently confined to these two papers (and the present one).

Such a situation seems in fact somewhat strange because the reversible contexts 1 and 2 are closely related. Destruction of resonant tori allows one to pass from context 1 to context 2 [38]: it is quite possible that $P \geqslant Q - n$ but $P < Q - (n - r)$. Nevertheless, break-up of resonant tori in reversible systems (for $r \leqslant n - 2$) has been studied by now only in the case where $P = Q = n$ [19, 42] and the inequality $P < Q - n + r$ is therefore never met. Excitation of elliptic normal modes allows one to pass from context 2 to context 1: it is quite possible that $P < Q - n$ but $P \geqslant Q - (n + r)$.

Remark 4. The statements "matrices L and $\mathcal{K}$ anti-commute" and "matrices L and $-\mathcal{K}$ anti-commute" are equivalent. Thus, on the level of linear operators, there is no difference between the reversible contexts 1 and 2.

All the discussion above has been devoted to autonomous flows (either Hamiltonian or reversible). The papers [38, 39] constituting the first steps in the reversible context 2 did not handle non-autonomous systems either. In the next section, we will examine reversible systems (within context 2) depending quasi-periodically on time. In context 1, such reversible systems were dealt with in [6, 22, 23, 35]. By the way, Moser's note [22] is the first paper on the reversible KAM theory whatsoever. Our paper [35] treats quasi-periodic perturbations in the reversible context 1, the Hamiltonian context, the volume preserving context, and the dissipative context from a unified viewpoint.

At the end of [38], we listed ten tentative topics and directions for further research. The eighth topic was non-autonomous perturbations depending on time periodically or quasi-periodically.

Note that families of reducible quasi-periodic invariant tori in KAM theory are Cantor-like because of resonances among the frequencies as well as of those between the frequencies and the imaginary parts of the Floquet exponents. Along an analytic subfamily of such tori, the frequencies and the imaginary parts of the Floquet exponents should therefore be constants (at least up to proportionality).

2. Quasi-periodic perturbations within the reversible context 2

For time-dependent vector fields on a manifold $\mathcal{M}$ equipped with an involution $G : \mathcal{M} \to \mathcal{M}$, Definition 2 of reversibility is modified as follows.

Definition 4. A time-dependent vector field V_t on $\mathcal{M}$ is said to be *reversible* with respect to involution G (or G-reversible) if $TG(V_t \circ G) \equiv -V_{-t}$ or, equivalently, if the function $t \mapsto G\big(a(-t)\big)$ is a solution of the equation $\dot{a} = V_t(a)$, $a \in \mathcal{M}$, whenever $t \mapsto a(t)$ is.

On $\mathcal{M}$, we will consider reversible systems $\dot{a} = V_t(a)$ depending on time quasi-periodically with $N \geqslant 1$ incommensurable basic frequencies $\Omega_1, \ldots, \Omega_N$:

$$V_t(a) \equiv \mathcal{V}(a, \Omega_1 t, \ldots, \Omega_N t),$$

where the function $\mathcal{V} = \mathcal{V}(a, X_1, \ldots, X_N)$ is 2π-periodic in each of the arguments $X_1, \ldots, X_N$. From the viewpoint of KAM theory, the natural problem is to look for quasi-periodic invariant tori of the corresponding autonomous system

$$\dot{a} = \mathcal{V}(a, X_1, \ldots, X_N), \quad \dot{X} = \Omega \tag{1}$$

on $\mathcal{M} \times \mathbb{T}^N$. Among the frequencies of such tori, there are N numbers $\Omega_1, \dots, \Omega_N$. It is easy to see that a non-autonomous system $\dot{a} = V_t(a)$ on $\mathcal{M}$ is reversible with respect to involution G if and only if the autonomous system (1) on $\mathcal{M} \times \mathbb{T}^N$ is reversible with respect to the involution $\mathcal{G} : (a, X) \mapsto \big(G(a), -X\big)$.

Now we are in the position to state the main result of this paper. We will consider analytic families of reducible quasi-periodic invariant tori in quasi-periodic perturbations of autonomous reversible systems. Let $x \in \mathbb{T}^n$, $y \in \mathcal{Y} \subset \mathbb{R}^m$, and $z \in \mathcal{O}_{2p}(0)$ be the phase space variables where $n \geqslant 0$, $m \geqslant 1$, $p \geqslant 0$, and $\mathcal{Y}$ is an open domain. The reversing involution is $G : (x, y, z) \mapsto (-x, -y, Kz)$ where $K \in \mathrm{GL}(2p, \mathbb{R})$ is an involutive matrix (K^2 is the $2p \times 2p$ identity matrix) with eigenvalues 1 and -1 of multiplicity p each. The domain $\mathcal{Y}$ in $\mathbb{R}^m$ where variable y ranges is assumed to contain the origin and to be invariant under the linear involution $y \mapsto -y$ (the reflection with respect to the origin). The neighborhood $\mathcal{O}_{2p}(0)$ where variable z ranges is supposed to be invariant under the linear involution $z \mapsto Kz$. The systems we will deal with depend

- on an angle variable $X \in \mathbb{T}^N$ ($N \geqslant 0$) subject to the equation $\dot{X} = \Omega$ with a fixed Diophantine vector $\Omega \in \mathbb{R}^N$ (according to the discussion above, for positive N this actually means a quasi-periodic dependence on time with frequency vector Ω),

- on an external parameter $\nu \in \mathcal{N} \subset \mathbb{R}^s$ ($s \geqslant 0$) where $\mathcal{N}$ is an open domain,

- and on a small perturbation parameter $\varepsilon \geqslant 0$.

These systems will be assumed to be reversible with respect to the involution $\mathcal{G} : (x, y, z, X) \mapsto (-x, -y, Kz, -X)$. We are interested in reducible quasi-periodic invariant $(n + N)$-tori in such systems in the "extended phase space" $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{O}_{2p}(0) \times \mathbb{T}^N$. It is clear that

$$P = \dim \mathrm{Fix}\, \mathcal{G} = p, \quad Q = \mathrm{codim}\, \mathrm{Fix}\, \mathcal{G} = n + m + p + N, \quad Q - (n + N) = m + p,$$

so that $P < Q - (n + N)$ since $m \geqslant 1$, and the situation pertains to the reversible context 2.

Consider a family of $\mathcal{G}$-reversible systems on $\mathbb{T}^n \times \mathcal{Y} \times \mathcal{O}_{2p}(0) \times \mathbb{T}^N$ of the form

$$
\begin{aligned}
\dot{x} &= H(y, \nu) + f^\sharp(x, y, z, \nu) + \varepsilon f(x, y, z, X, \nu, \varepsilon), \\
\dot{y} &= \Xi(y, \nu) + g^\sharp(x, y, z, \nu) + \varepsilon g(x, y, z, X, \nu, \varepsilon), \\
\dot{z} &= \Lambda(y, \nu)z + h^\sharp(x, y, z, \nu) + \varepsilon h(x, y, z, X, \nu, \varepsilon), \\
\dot{X} &= \Omega
\end{aligned}
\tag{2}
$$

(with a $2p \times 2p$ matrix-valued function Λ), where $f^\sharp = \mathrm{O}\big(|z|\big)$, $g^\sharp = \mathrm{O}\big(|z|^2\big)$, and $h^\sharp = \mathrm{O}\big(|z|^2\big)$. Reversibility of (2) with respect to $\mathcal{G}$ means that

$$
H(-y, \nu) \equiv H(y, \nu), \quad \Xi(-y, \nu) \equiv \Xi(y, \nu),
$$
$$
\Lambda(-y, \nu)K \equiv -K\Lambda(y, \nu)
$$

and

$$
\begin{aligned}
f^\sharp(-x, -y, Kz, \nu) &\equiv f^\sharp(x, y, z, \nu), & f(-x, -y, Kz, -X, \nu, \varepsilon) &\equiv f(x, y, z, X, \nu, \varepsilon), \\
g^\sharp(-x, -y, Kz, \nu) &\equiv g^\sharp(x, y, z, \nu), & g(-x, -y, Kz, -X, \nu, \varepsilon) &\equiv g(x, y, z, X, \nu, \varepsilon), \\
h^\sharp(-x, -y, Kz, \nu) &\equiv -Kh^\sharp(x, y, z, \nu), & h(-x, -y, Kz, -X, \nu, \varepsilon) &\equiv -Kh(x, y, z, X, \nu, \varepsilon).
\end{aligned}
$$

All the functions H, Ξ, Λ, $f^\sharp$, $g^\sharp$, $h^\sharp$, f, g, and h are assumed to be analytic in all their arguments.

The eigenvalues of the matrix $\Lambda(0,\nu)$ anti-commuting with K come in pairs $\lambda, -\lambda$ for any $\nu \in \mathcal{N}$ [27, 29]. Let the spectrum of $\Lambda(0,\nu)$ be *simple* for any $\nu \in \mathcal{N}$ and have the form

$$
\begin{aligned}
&\pm\alpha_1(\nu), \ldots, \pm\alpha_{d_1}(\nu), \qquad \pm\mathrm{i}\beta_1(\nu), \ldots, \pm\mathrm{i}\beta_{d_2}(\nu), \\
&\pm\alpha_{d_1+1}(\nu) \pm \mathrm{i}\beta_{d_2+1}(\nu), \ldots, \pm\alpha_{d_1+d_3}(\nu) \pm \mathrm{i}\beta_{d_2+d_3}(\nu),
\end{aligned}
\tag{3}
$$

where the numbers $d_1 \geqslant 0$, $d_2 \geqslant 0$, $d_3 \geqslant 0$ do not depend on ν ($d_1 + d_2 + 2d_3 = p$), $\alpha_k(\nu) > 0$ for all $1 \leqslant k \leqslant d_1 + d_3$, $\nu \in \mathcal{N}$, and $\beta_l(\nu) > 0$ for all $1 \leqslant l \leqslant d_2 + d_3$, $\nu \in \mathcal{N}$.

Fix an *arbitrary* (possibly, empty) subset of indices

$$
\mathfrak{Z} \subset \{1; 2; \ldots; d_1 + d_3\}
$$

consisting of κ elements ($0 \leqslant \kappa \leqslant d_1 + d_3$).

Theorem 1. *Suppose that*

- $s \geqslant n + m + d_2 + d_3 + \kappa$,

- $\Xi(0, \nu^0) = 0$ *for some* $\nu^0 \in \mathcal{N}$ *[we will use the notation* $\omega = H(0, \nu^0) \in \mathbb{R}^n$, $\alpha^0 = \alpha(\nu^0) \in \mathbb{R}^{d_1+d_3}$, $\beta^0 = \beta(\nu^0) \in \mathbb{R}^{d_2+d_3}$*]*,

- *the vectors* ω, Ω, *and* β^0 *satisfy the following Diophantine condition: there exist constants* $\tau > n + N - 1$ *and* $\gamma > 0$ *such that*

$$
\left| \langle j, \omega \rangle + \langle J, \Omega \rangle + \langle q, \beta^0 \rangle \right| \geqslant \gamma \big(|j| + |J| \big)^{-\tau}
$$

  for all $(j, J) \in \mathbb{Z}^{n+N} \setminus \{0\}$ *and* $q \in \mathbb{Z}^{d_2+d_3}$, $|q| = |q_1| + \cdots + |q_{d_2+d_3}| \leqslant 2$,

- *the mapping from* $\mathcal{N}$ *to* $\mathbb{R}^{n+m+d_2+d_3+\kappa}$ *given by*

$$
\nu \mapsto \big(H(0, \nu), \Xi(0, \nu), \beta(\nu), \alpha_k(\nu), k \in \mathfrak{Z} \big)
\tag{4}
$$

  is submersive *at point* ν^0, *i.e.,*

$$
\mathrm{rank}\, \frac{\partial \big(H(0, \nu), \Xi(0, \nu), \beta(\nu), \alpha_k(\nu), k \in \mathfrak{Z} \big)}{\partial \nu} \bigg|_{\nu=\nu^0} = n + m + d_2 + d_3 + \kappa.
$$

Then for sufficiently small ε *there exists an* $(s-n-m-d_2-d_3-\kappa)$*-dimensional analytic surface* $\mathcal{S}_\varepsilon \subset \mathcal{N}$ *such that for any* $\nu \in \mathcal{S}_\varepsilon$, *system (2) admits an analytic reducible invariant* $(n+N)$*-torus carrying Diophantine quasi-periodic motions with frequency vector* (ω, Ω). *The Floquet exponents of this torus are*

$$
\begin{aligned}
&\underbrace{0, \ldots, 0}_{m}, \qquad \pm\alpha_1'(\nu, \varepsilon), \ldots, \pm\alpha_{d_1}'(\nu, \varepsilon), \qquad \pm\mathrm{i}\beta_1^0, \ldots, \pm\mathrm{i}\beta_{d_2}^0, \\
&\pm\alpha_{d_1+1}'(\nu, \varepsilon) \pm \mathrm{i}\beta_{d_2+1}^0, \ldots, \pm\alpha_{d_1+d_3}'(\nu, \varepsilon) \pm \mathrm{i}\beta_{d_2+d_3}^0
\end{aligned}
\tag{5}
$$

[cf. (3)], where $\alpha_k'(\nu, \varepsilon) > 0$ *for all* $1 \leqslant k \leqslant d_1 + d_3$, $\nu \in \mathcal{S}_\varepsilon$, *and* $\alpha_k'(\nu, \varepsilon) \equiv \alpha_k^0$ *for* $k \in \mathfrak{Z}$. *These tori and the numbers* $\alpha_k'(\nu, \varepsilon)$, $k \notin \mathfrak{Z}$, *depend analytically on* $\nu \in \mathcal{S}_\varepsilon$ *and on* $\varepsilon^{1/2}$. *At* $\varepsilon = 0$, *the surface* $\mathcal{S}_0$ *contains* ν^0 *and all the tori are* $\{y = 0, z = 0\}$.

This theorem describes the persistence of the unperturbed reducible invariant torus $\{y = 0,\, z = 0;\, \nu = \nu^0\}$ with the preservation of

- the frequencies $\omega_1, \ldots, \omega_n$,

- all the imaginary parts $\pm\beta_1^0, \ldots, \pm\beta_{d_2+d_3}^0$ of the Floquet exponents,

- and an arbitrary subcollection (of length κ) of the pairs of the real parts $\pm\alpha_1^0, \ldots, \pm\alpha_{d_1+d_3}^0$ of the Floquet exponents.

The situation resembles the so called *partial preservation of Floquet exponents* [36] in the "well developed" autonomous contexts of KAM theory (the reversible context 1, the Hamiltonian context, the volume preserving context, and the dissipative context).

The autonomous case of Theorem 1 ($N = 0$) is the main result of our previous paper [39]. It is tempting to reduce Theorem 1 with $N > 0$ to this particular case via regarding (x, X) as a new variable x and $n + N$ as a new dimension n. Unfortunately, such an attempt would lead to the mapping

$$\nu \mapsto \big(H(0, \nu),\, \Omega,\, \Xi(0, \nu),\, \beta(\nu),\, \alpha_k(\nu), k \in \mathfrak{Z}\big) \tag{6}$$

[cf. (4)] which cannot be submersive because its second component is a constant. Moreover, in any case, this approach would require $s \geqslant n + N + m + d_2 + d_3 + \kappa$.

However, one really can reduce Theorem 1 with arbitrary N to its particular case $N = 0$. Proofs in KAM theory are generally believed to be very complicated and tedious. Nevertheless, this theory also includes many powerful methods that enable one to deduce various statements from simpler ones in a very straightforward manner. Here are some examples (the references we give just illustrate the methods in question).

- KAM-type theorems for vector fields and for diffeomorphisms [14].

- A reduction of "local" theorems (concerning invariant tori near equilibria or closed trajectories) to "global" ones [27].

- Easy proofs of excitation of elliptic normal modes using "conventional" theorems with Rüssmann-like nondegeneracy conditions [10, 31].

- Herman's method of reducing KAM theorems with weak nondegeneracy conditions to theorems with nondegeneracy conditions of the submersivity type [9, 10, 12, 31, 34–36].

- The easy proofs of the autonomous case of Theorem 1 (and some other results in the reversible KAM theory) employing Moser's modifying terms theorem [24, 38, 39].

A reduction of Theorem 1 with positive N to the autonomous case is a one more example (probably the simplest one). Introduce an artificial additional external parameter $\Theta \in \mathcal{O}_N(0)$ and replace the last equation $\dot{X} = \Omega$ in systems (2) with the equation $\dot{X} = \Omega + \Theta$. This does lead to the autonomous framework with

- (x, X) playing the role of x,

- $n + N$ playing the role of n,

- (ω, Ω) playing the role of ω,

- (ν, Θ) playing the role of ν,

- $s + N$ playing the role of s,

- $\mathcal{N} \times \mathcal{O}_N(0)$ playing the role of $\mathcal{N}$,

- $(\nu^0, 0) \in \mathbb{R}^{s+N}$ playing the role of ν^0.

Indeed, instead of (4), one now has to consider the mapping from $\mathcal{N} \times \mathcal{O}_N(0)$ to $\mathbb{R}^{n+N+m+d_2+d_3+\kappa}$ given by

$$(\nu, \Theta) \mapsto \big(H(0, \nu),\, \Omega + \Theta,\, \Xi(0, \nu),\, \beta(\nu),\, \alpha_k(\nu), k \in \mathfrak{Z}\big) \tag{7}$$

[cf. (6)]. Since the mapping (4) is submersive at point $\nu = \nu^0$, the mapping (7) is submersive at point $\nu = \nu^0$, $\Theta = 0$.

Apply the autonomous Theorem 1 proven in [39] using Moser's modifying terms theorem [24]. We arrive at the conclusion that for sufficiently small ε, there exists an $(s - n - m - d_2 - d_3 - \kappa)$-dimensional analytic surface $\mathcal{S}_\varepsilon \subset \big(\mathcal{N} \times \mathcal{O}_N(0)\big)$ such that for any $(\nu, \Theta) \in \mathcal{S}_\varepsilon$, the modified system (2) [with $\dot{X} = \Omega + \Theta$ instead of $\dot{X} = \Omega$] admits an analytic reducible invariant $(n + N)$-torus carrying Diophantine quasi-periodic motions with frequency vector (ω, Ω). The Floquet exponents of this torus are

$$\underbrace{0, \ldots, 0}_{m}, \qquad \pm\alpha_1'(\nu, \Theta, \varepsilon), \ldots, \pm\alpha_{d_1}'(\nu, \Theta, \varepsilon), \qquad \pm\mathrm{i}\beta_1^0, \ldots, \pm\mathrm{i}\beta_{d_2}^0,$$
$$\pm\alpha_{d_1+1}'(\nu, \Theta, \varepsilon) \pm \mathrm{i}\beta_{d_2+1}^0, \ldots, \pm\alpha_{d_1+d_3}'(\nu, \Theta, \varepsilon) \pm \mathrm{i}\beta_{d_2+d_3}^0 \tag{8}$$

[cf. (5)], where $\alpha_k'(\nu, \Theta, \varepsilon) > 0$ for all $1 \leqslant k \leqslant d_1 + d_3$, $(\nu, \Theta) \in \mathcal{S}_\varepsilon$, and $\alpha_k'(\nu, \Theta, \varepsilon) \equiv \alpha_k^0$ for $k \in \mathfrak{Z}$. These tori and the numbers $\alpha_k'(\nu, \Theta, \varepsilon)$, $k \notin \mathfrak{Z}$, depend analytically on $(\nu, \Theta) \in \mathcal{S}_\varepsilon$ and on $\varepsilon^{1/2}$. At $\varepsilon = 0$, the surface $\mathcal{S}_0$ contains the point $(\nu^0, 0)$ and all the tori are $\{y = 0, z = 0\}$.

Now it suffices to verify that the surface $\mathcal{S}_\varepsilon$ lies in fact in $\mathcal{N} \times \{0\}$ for each ε, i.e. that $\Theta = 0$ on $\mathcal{S}_\varepsilon$ [and the dependence of α' on Θ in (8) is dummy]. Consider the $(n+N)$-torus $\mathcal{T}$ corresponding to an arbitrary point $(\nu, \Theta) \in \mathcal{S}_\varepsilon$. According to the standard reflection lemma of Section 1.2, one can introduce in $\mathcal{T}$ a coordinate frame $\varphi \in \mathbb{T}^n$, $\Phi \in \mathbb{T}^N$ in which the dynamics on $\mathcal{T}$ takes the form

$$\dot{\varphi} = \omega, \quad \dot{\Phi} = \Omega \tag{9}$$

and the restriction of the involution $\mathcal{G} : (x, X, y, z) \mapsto (-x, -X, -y, Kz)$ to $\mathcal{T}$ takes the form

$$\mathcal{G}|_\mathcal{T} : (\varphi, \Phi) \mapsto (-\varphi, -\Phi). \tag{10}$$

The torus $\mathcal{T}$ is close to the unperturbed $(n + N)$-torus $\{y = 0, z = 0\}$ at $\nu = \nu^0$, $\Theta = 0$, $\varepsilon = 0$ for which one can set $\varphi = x$, $\Phi = X$. Consequently, the coordinates φ, Φ in $\mathcal{T}$ can be chosen in such a way that the torus $\mathcal{T}$ will be given by the equations

$$x = \varphi + A(\varphi, \Phi), \quad X = \Phi + B(\varphi, \Phi), \quad y = C(\varphi, \Phi), \quad z = D(\varphi, \Phi),$$

A, B, C, and D being small analytic functions with values in $\mathbb{R}^n$, $\mathbb{R}^N$, $\mathbb{R}^m$, and $\mathbb{R}^{2p}$, respectively. In view of (10) these functions satisfy the identities

$$A(-\varphi, -\Phi) \equiv -A(\varphi, \Phi), \qquad\qquad B(-\varphi, -\Phi) \equiv -B(\varphi, \Phi),$$
$$C(-\varphi, -\Phi) \equiv -C(\varphi, \Phi), \qquad\qquad D(-\varphi, -\Phi) \equiv KD(\varphi, \Phi).$$

Differentiating the relation $X = \Phi + B(\varphi, \Phi)$ with respect to time and taking (9) and the equation $\dot{X} = \Omega + \Theta$ into account, one obtains

$$\Omega + \Theta \equiv \Omega + \frac{\partial B}{\partial \varphi}\omega + \frac{\partial B}{\partial \Phi}\Omega,$$

i.e.,

$$\frac{\partial B}{\partial \varphi}\omega + \frac{\partial B}{\partial \Phi}\Omega \equiv \Theta. \tag{11}$$

The mean value of the left-hand side of (11) over $\mathbb{T}^{n+N}$ vanishes, whence $\Theta = 0$. Moreover, the identity (11) with $\Theta = 0$ and the fact that the numbers $\omega_1, \ldots, \omega_n, \Omega_1, \ldots \Omega_N$ are independent over rationals imply easily that $B = \mathrm{const}$ (cf. [35, section 6.2]). Since the function B is odd, $B \equiv 0$. The proof of Theorem 1 with arbitrary $N \geqslant 0$ is completed.

Remark finally that the reversible KAM theory can be extended to systems quasi-periodic not only in time but also in some spatial variables [15].

Acknowledgments

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